

THE MAXIMAL OPERATORS OF LOGARITHMIC MEANS OF ONE-DIMENSIONAL VILENKIN-FOURIER SERIES.

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ABSTRACT. The main aim of this paper is to investigate (H_p, L_p) -type inequalities for maximal operators of logarithmic means of one-dimensional Vilenkin-Fourier series.

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1. INTRODUCTION

In one-dimensional case the weak type inequality

$$\mu(\sigma^* f > \lambda) \leq \frac{c}{\lambda} \|f\|_1 \quad (\lambda > 0)$$

can be found in Zygmund [20] for the trigonometric series, in Schipp [11] for Walsh series and in Pál, Simon [10] for bounded Vilenkin series. Again in one-dimensional, Fujji [3] and Simon [13] verified that σ^* is bounded from H_1 to L_1 . Weisz [17] generalized this result and proved the boundedness of σ^* from the martingale space H_p to the space L_p for $p > 1/2$. Simon [12] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ due to Goginava ([7], see also [2]).

Riesz's logarithmic means with respect to the trigonometric system was studied by a lot of authors. We mentioned, for instance, the paper by Szasz [14] and Yabuta [19]. This means with respect to the Walsh and Vilenkin systems by Simon [12] and Gát [4].

Móricz and Siddiqi [9] investigate the approximation properties of some special Nörlund means of Walsh-Fourier series of L_p function in norm. The case when $q_k = 1/k$ is excluded, since the methods of Móricz and Siddiqi are not applicable to Nörlund logarithmic means. In [5] Gát and Goginava proved some convergence and divergence properties of the Nörlund logarithmic means of functions in the class of continuous functions and in the Lebesgue space L_1 . Among there, they gave a negative answer to the question of Móricz and Siddiqi [9]. Gát and Goginava [6] proved that for each measurable function $\phi(u) = o(u\sqrt{\log u})$ there exists an integrable function f , such that

$$\int_{G_m} \phi(|f(x)|) d\mu(x) < \infty$$

and there exist a set with positive measure, such that the Walsh-logarithmic means of the function diverge on this set.

The main aim of this paper is to investigate (H_p, L_p) -type inequalities for the maximal operators of Riesz and Nörlund logarithmic means of one-dimensional Vilenkin-Fourier series. We prove that the maximal operator R^* is bounded from the Hardy space H_p to the space

L_p when $p > 1/2$. We also shows that when $0 < p \leq 1/2$ there exists a martingale $f \in H_p$, for which

$$\|R^*f\|_{L_p} = +\infty.$$

For the Nörlund logarithmic means we prove that when $0 < p \leq 1$ there exists a martingale $f \in H_p$ for which

$$\|L^*f\|_{L_p} = +\infty.$$

Analogical theorems for Walsh-Paley system is proved in [8].

2. DEFINITIONS AND NOTATIONS

Let N_+ denote the set of the positive integers, $N := N_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ denote a sequence of the positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \dots, m_k - 1\}$ the addition group of integers modulo m_k .

Define the group G_m as the complete direct product of the groups Z_{m_i} with the product of the discrete topologies of Z_{m_j} 's.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k, \quad (j \in Z_{m_k})$$

is the Haar measure on G_{m_k} , with $\mu(G_m) = 1$.

If $\sup_n m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded then G_m is said to be an unbounded Vilenkin group. **In this paper we discuss bounded Vilenkin groups only.**

The elements of G_m represented by sequences

$$x := (x_0, x_1, \dots, x_j, \dots), \quad (x_i \in Z_{m_j}).$$

It is easy to give a base for the neighborhood of G_m

$$\begin{aligned} I_0(x) &:= G_m, \\ I_n(x) &:= \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}, \quad (x \in G_m, n \in N). \end{aligned}$$

Denote $I_n := I_n(0)$, for $n \in N_+$.

If we define the so-called generalized number system based on m in the following way :

$$M_0 := 1, \quad M_{k+1} := m_k M_k, \quad (k \in N),$$

then every $n \in N$, can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$, ($j \in N_+$) and only a finite number of n_j 's differ from zero.

Next, we introduce on G_m an ortonormal system which is called the Vilenkin system. At first define the complex valued function $r_k(x) : G_m \rightarrow C$, The generalized Rademacher functions as

$$r_k(x) := \exp(2\pi i x_k / m_k), \quad (i^2 = -1, x \in G_m, k \in N).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in N)$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in N).$$

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$. [1, 15]

Now we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet kernels with respect to the Vilenkin system ψ in the usual manner:

$$\begin{aligned} \widehat{f}(k) &:= \int_{G_m} f \bar{\psi}_k d\mu, & (k \in N), \\ S_n f &:= \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, & (n \in N_+, S_0 f := 0), \\ \sigma_n f &:= \frac{1}{n} \sum_{k=0}^{n-1} S_k f, & (n \in N_+), \\ D_n &:= \sum_{k=0}^{n-1} \psi_k, & (n \in N_+). \end{aligned}$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases}$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}, \quad (0 < p < \infty).$$

The σ -algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by F_n ($n \in N$). Denote by $f = (f^{(n)}, n \in N)$ a martingale with respect to F_n ($n \in N$). (for details see e.g. [16]).

The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in N} |f^{(n)}|.$$

In case $f \in L_1(G_m)$, the maximal functions are also be given by

$$f^*(x) = \sup_{n \in N} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u) d\mu(u) \right|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingale for which

$$\|f\|_{H_p} := \|f^*\|_{L_p} < \infty.$$

If $f \in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}(f) : n \in N)$ is a martingale.

If $f = (f^{(n)}, n \in N)$ is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \rightarrow \infty} \int_{G_m} f^{(k)}(x) \overline{\Psi_i}(x) d\mu(x).$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f) : n \in N)$ obtained from f .

In the literature, there is the notion of Riesz' s logarithmic means of the Fourier series. The n -th Riesz' s logarithmic means of the Fourier series of an integrable function f is defined by

$$R_n f(x) := \frac{1}{l_n} \sum_{k=1}^n \frac{S_k f(x)}{k},$$

where

$$l_n := \sum_{k=1}^n (1/k).$$

Let $\{q_k : k > 0\}$ be a sequence of nonnegative numbers. The n -th Nörlund means for the Fourier series of f is defined by

$$\frac{1}{Q_n} \sum_{k=1}^n q_{n-k} S_k f,$$

where

$$Q_n := \sum_{k=1}^n q_k.$$

If $q_k = k$, then we get Nörlund logarithmic means

$$L_n f(x) := \frac{1}{l_n} \sum_{k=1}^n \frac{S_k f(x)}{n-k}.$$

It is a kind of "reverse" Riesz' s logarithmic means.

In this paper we call this means logarithmic means.

For the martingale f we consider the following maximal operators of

$$\begin{aligned}
R^* f(x) &: = \sup_{n \in N} |R_n f(x)|, \\
L^* f(x) &: = \sup_{n \in N} |L_n f(x)|, \\
\sigma^* f(x) &: = \sup_{n \in N} |\sigma_n f(x)|.
\end{aligned}$$

A bounded measurable function a is p -atom, if there exists a dyadic interval I , such that

$$\begin{cases} a) & \int_I a d\mu = 0, \\ b) & \|a\|_\infty \leq \mu(I)^{-1/p}, \\ c) & \text{supp}(a) \subset I. \end{cases}$$

3. FORMULATION OF MAIN RESULT

Theorem 1. *Let $p > 1/2$. Then the maximal operator R^* is bounded from the Hardy space H_p to the space L_p .*

Theorem 2. *Let $0 < p \leq 1/2$. Then there exists a martingale $f \in H_p$ such that*

$$\|R^* f\|_p = +\infty.$$

Corollary 1. *Let $0 < p \leq 1/2$. Then there exists a martingale $f \in H_p$ such that*

$$\|\sigma^* f\|_p = +\infty.$$

Theorem 3. *Let $0 < p \leq 1$. Then there exists a martingale $f \in L_p$ such that*

$$\|L^* f\|_p = +\infty.$$

4. AUXILIARY PROPOSITIONS

Lemma 1. [18] *A martingale $f = (f^{(n)}, n \in N)$ is in H_p ($0 < p \leq 1$) if and only if there exist a sequence $(a_k, k \in N)$ of p -atoms and a sequence $(\mu_k, k \in N)$ of a real numbers such that for every $n \in N$:*

$$(1) \quad \sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f^{(n)},$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{K=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decomposition of f of the form (1).

5. PROOF OF THE THEOREM

Proof of theorem 1. Using Abel transformation we obtain

$$R_n f(x) = \frac{1}{l_n} \sum_{j=1}^{n-1} \frac{\sigma_j f(x)}{j+1} + \frac{\sigma_n f(x)}{l_n},$$

Consequently,

$$(2) \quad L^* f \leq c \sigma^* f.$$

On the other hand Weisz[17] proved that σ^* is bounded from the Hardy space H_p to the space L_p when $p > 1/2$. Hence, from (2) we conclude that R^* is bounded from the martingale Hardy space H_p to the space L_p when $p > 1/2$.

Proof of theorem 2. Let $\{\alpha_k : k \in N\}$ be an increasing sequence of the positive integers such that

$$(3) \quad \sum_{k=0}^{\infty} \alpha_k^{-p/2} < \infty,$$

$$(4) \quad \sum_{\eta=0}^{k-1} \frac{(M_{2\alpha_\eta})^{1/p}}{\sqrt{\alpha_\eta}} < \frac{(M_{2\alpha_k})^{1/p}}{\sqrt{\alpha_k}},$$

$$(5) \quad \frac{(M_{2\alpha_{k-1}})^{1/p}}{\sqrt{\alpha_{k-1}}} < \frac{M_{\alpha_k}}{\alpha_k^{3/2}}.$$

We note that such an increasing sequence $\{\alpha_k : k \in N\}$ which satisfies conditions (3)-(5) can be constructed.

Let

$$f^{(A)}(x) = \sum_{\{k; 2\alpha_k < A\}} \lambda_k a_k,$$

where

$$\lambda_k = \frac{m_{2\alpha_k}}{\sqrt{\alpha_k}}$$

and

$$a_k(x) = \frac{M_{2\alpha_k}^{1/p-1}}{m_{2\alpha_k}} \left(D_{M_{2\alpha_k+1}}(x) - D_{M_{2\alpha_k}}(x) \right).$$

It is easy to show that

$$\begin{aligned} \|a_k\|_\infty &\leq \frac{M_{2\alpha_k}^{1/p-1}}{m_{2\alpha_k}} M_{2\alpha_k+1} \\ &\leq (M_{2\alpha_k})^{1/p} = (\text{supp}(a_k))^{-1/p}, \end{aligned}$$

$$(6) \quad S_{M_A} a_k(x) = \begin{cases} a_k(x), & 2\alpha_k < A, \\ 0, & 2\alpha_k \geq A. \end{cases}$$

$$\begin{aligned} f^{(A)}(x) &= \sum_{\{k; 2\alpha_k < A\}} \lambda_k a_k = \sum_{k=0}^{\infty} \lambda_k S_{M_A} a_k(x), \\ \text{supp}(a_k) &= I_{2\alpha_k}, \\ \int_{I_{2\alpha_k}} a_k d\mu &= 0. \end{aligned}$$

from (3) and lemma 1 we conclude that $f = (f^{(n)}, n \in N) \in H_p$.

Let

$$q_A^s = M_{2A} + M_{2s} - 1, \quad A > S.$$

Then we can write

$$\begin{aligned} (7) \quad R_{q_{\alpha_k}^s} f(x) &= \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=1}^{q_{\alpha_k}^s} \frac{S_j f(x)}{j} \\ &= \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{S_j f(x)}{j} \\ &\quad + \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s} \frac{S_j f(x)}{j} \\ &= I + II. \end{aligned}$$

It is easy to show that

$$(8) \quad \widehat{f}(j) = \begin{cases} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}}, & \text{if } j \in \{M_{2\alpha_k}, \dots, M_{2\alpha_k+1} - 1\}, \quad k = 0, 1, 2, \dots, \\ 0, & \text{if } j \notin \bigcup_{k=1}^{\infty} \{M_{2\alpha_k}, \dots, M_{2\alpha_k+1} - 1\}. \end{cases}$$

Let $j < M_{2\alpha_k}$. Then from (4) and (8) we have

$$\begin{aligned}
(9) \quad & |S_j f(x)| \\
& \leq \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_\eta+1}-1} |\widehat{f}(v)| \\
& \leq \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_\eta+1}-1} \frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}} \\
& \leq c \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p}}{\sqrt{\alpha_\eta}} \leq \frac{c M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}.
\end{aligned}$$

Consequently

$$\begin{aligned}
(10) \quad |I| & \leq \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{|S_j f(x)|}{j} \\
& \leq \frac{c}{\alpha_k} \frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{1}{j} \\
& \quad \frac{M_{2\alpha_{k-1}}^{1/p}}{c \sqrt{\alpha_{k-1}}}.
\end{aligned}$$

Let $M_{2\alpha_k} \leq j \leq q_{\alpha_k}^s$. Then we have the following

$$\begin{aligned}
(11) \quad S_j f(x) &= \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_\eta+1}-1} \widehat{f}(v) \psi_v(x) + \sum_{v=M_{2\alpha_k}}^{j-1} \widehat{f}(v) \psi_v(x) \\
&= \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}} \left(D_{M_{2\alpha_\eta+1}}(x) - D_{M_{2\alpha_\eta}}(x) \right) \\
& \quad + \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \left(D_j(x) - D_{M_{2\alpha_k}}(x) \right).
\end{aligned}$$

This gives that

$$\begin{aligned}
(12) \quad II &= \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s} \frac{1}{j} \left(\sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}} \left(D_{M_{2\alpha_\eta+1}}(x) - D_{M_{2\alpha_\eta}}(x) \right) \right) \\
& \quad + \frac{1}{l_{q_{\alpha_k}^s}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s} \frac{\left(D_j(x) - D_{M_{2\alpha_k}}(x) \right)}{j} \\
&= II_1 + II_2.
\end{aligned}$$

To discuss II_1 , we use (4). Thus we can write:

$$(13) \quad |II_1| \leq c \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p}}{\sqrt{\alpha_\eta}} \leq \frac{cM_{2\alpha_{k-1}}}{\sqrt{\alpha_{k-1}}}.$$

Since

$$(14) \quad D_{j+M_{2\alpha_k}}(x) = D_{M_{2\alpha_k}}(x) + \psi_{M_{2\alpha_k}}(x) D_j(x), \quad \text{when } j < M_{2\alpha_k},$$

for II_2 we have

$$(15) \quad \begin{aligned} II_2 &= \frac{1}{l_{q_{\alpha_k}^s}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \sum_{j=0}^{M_{2s}} \frac{D_{j+M_{2\alpha_k}}(x) - D_{M_{2\alpha_k}}(x)}{j+M_{2\alpha_k}} \\ &= \frac{1}{l_{q_{\alpha_k}^s}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \psi_{M_{2\alpha_k}} \sum_{j=0}^{M_{2s}-1} \frac{D_j(x)}{j+M_{2\alpha_k}}. \end{aligned}$$

We write

$$R_{q_{\alpha_k}^s} f(x) = I + II_1 + II_2,$$

Then by (5), (7), (10) and (12)-(15) we have

$$\begin{aligned} \left| R_{q_{\alpha_k}^s} f(x) \right| &\geq |II_2| - |I| - |II_1| \\ &\geq |II_2| - c \frac{M_{\alpha_k}}{\alpha_k^{3/2}} \\ &\geq \frac{c}{\alpha_k} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \left| \sum_{j=0}^{M_{2s}-1} \frac{D_j(x)}{j+M_{2\alpha_k}} \right| - c \frac{M_{\alpha_k}}{\alpha_k^{3/2}}. \end{aligned}$$

Let $0 < p \leq 1/2$, $x \in I_{2s} \setminus I_{2s+1}$ for $s = [2\alpha_k/3], \dots, \alpha_k$. Then it is evident

$$\left| \sum_{j=0}^{M_{2s}-1} \frac{D_j(x)}{j+M_{2\alpha_k}} \right| \geq \frac{cM_{2s}^2}{M_{2\alpha_k}}.$$

Hence we can write

$$\begin{aligned}
\left| R_{q_{\alpha_k}^s} f(x) \right| &\geq \frac{c}{\alpha_k} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \frac{cM_{2s}^2}{M_{2\alpha_k}} - c \frac{M_{\alpha_k}}{\alpha_k^{3/2}} \\
&\geq \frac{cM_{2\alpha_k}^{1/p-2} M_{2s}^2}{\alpha_k^{3/2}} - c \frac{M_{\alpha_k}}{\alpha_k^{3/2}} \\
&\geq \frac{cM_{2\alpha_k}^{1/p-2} M_{2s}^2}{\alpha_k^{3/2}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\int_{G_m} |R^* f(x)|^p d\mu(x) \\
&\geq \sum_{s=[2\alpha_k/3]}^{\alpha_k} \int_{I_{2s} \setminus I_{2s+1}} \left| R_{q_{\alpha_k}^s} f(x) \right|^p d\mu(x) \\
&\geq \sum_{s=[2\alpha_k/3]}^{\alpha_k} \int_{I_{2s} \setminus I_{2s+1}} \left(\frac{cM_{2\alpha_k}^{1/p-2} M_{2s}^2}{\alpha_k^{3/2}} \right)^p d\mu(x) \\
&\geq c \sum_{s=[2\alpha_k/3]}^{\alpha_k} \frac{M_{2\alpha_k}^{1-2p} M_{2s}^{2p-1}}{\alpha_k^{3p/2}} \\
&\geq \begin{cases} \frac{2^{\alpha_k(1-2p)}}{\alpha_k^{3p/2}}, & \text{when } 0 < p < 1/2, \\ c\alpha_k^{1/4}, & \text{when } p = 1/2, \end{cases} \rightarrow \infty, \quad \text{when } k \rightarrow \infty.
\end{aligned}$$

which complete the proof of the theorem 2.

Proof of theorem 3. We write

$$\begin{aligned}
(16) \quad L_{q_{\alpha_k}^s} f(x) &= \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=1}^{q_{\alpha_k}^s} \frac{S_j f(x)}{q_{\alpha_k}^s - j} \\
&= \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{S_j f(x)}{q_{\alpha_k}^s - j} \\
&\quad + \frac{1}{q_{\alpha_k}^s} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s} \frac{S_j f(x)}{q_{\alpha_k}^s - j} \\
&= III + IV.
\end{aligned}$$

Since (see 9)

$$|S_j f(x)| \leq c \frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}, \quad j < M_{2\alpha_k}.$$

For III we can write

$$(17) \quad |III| \leq \frac{c}{\alpha_k} \sum_{j=0}^{M_{2\alpha_k-1}} \frac{1}{q_{\alpha_k}^s - j} \frac{M_{2\alpha_k-1}^{1/p}}{\sqrt{\alpha_{k-1}}} \leq c \frac{M_{2\alpha_k-1}^{1/p}}{\sqrt{\alpha_{k-1}}}.$$

Using (11) we have

$$(18) \quad \begin{aligned} IV &= \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s} \frac{1}{q_{\alpha_k,s} - j} \left(\sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}} \left(D_{M_{2\alpha_\eta+1}}(x) - D_{M_{2\alpha_\eta}}(x) \right) \right) \\ &\quad + \frac{1}{l_{q_{\alpha_k}^s}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s} \frac{\left(D_j(x) - D_{M_{2\alpha_k}}(x) \right)}{q_{\alpha_k}^s - j} \\ &= IV_1 + IV_2. \end{aligned}$$

Applying (4) in IV_1 we have

$$(19) \quad |IV_1| \leq c \frac{M_{2\alpha_k-1}^{1/p}}{\sqrt{\alpha_{k-1}}}.$$

From (14) we obtain

$$(20) \quad IV_2 = \frac{1}{l_{q_{\alpha_k,s}}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \psi_{M_{2\alpha_k}} \sum_{j=0}^{M_{2s}-1} \frac{D_j(x)}{M_{2s} - j}.$$

Let $x \in I_{2s} \setminus I_{2s+1}$. Then $D_j(x) = j$, $j < M_{2s}$. Consequently

$$\begin{aligned} &\sum_{j=0}^{M_{2s}-1} \frac{D_j(x)}{M_{2s} - j} = \sum_{j=0}^{M_{2s}-1} \frac{j}{M_{2s} - j} \\ &= \sum_{j=0}^{M_{2s}-1} \left(\frac{M_{2s}}{M_{2s} - j} - 1 \right) \geq csM_{2s}. \end{aligned}$$

Then

$$(21) \quad |IV_2| \geq c \frac{M_{2\alpha_k-1}^{1/p-1}}{\alpha_k^{3/2}} s M_{2s}, \quad x \in I_{2s} \setminus I_{2s+1}.$$

Combining (5), (16)-(21) for $x \in I_{2s} \setminus I_{2s+1}$, $s = [2\alpha_k/3] \dots \alpha_k$ and $0 < p \leq 1$ we have

$$\begin{aligned}
& \left| L_{q_{\alpha_k}^s} f(x) \right| \\
& \geq c \frac{M_{2\alpha_k-1}^{1/p-1}}{\alpha_k^{3/2}} s M_{2s} - c \frac{M_{\alpha_k}}{\alpha_k^{3/2}} \\
& \geq c \frac{M_{2\alpha_k-1}^{1/p-1}}{\alpha_k^{3/2}} s M_{2s}.
\end{aligned}$$

Then

$$\begin{aligned}
& \int_{G_m} |L^* f(x)|^p d\mu(x) \\
& \geq \sum_{s=[2\alpha_k/3]}^{m_k} \int_{I_{2s} \setminus I_{2s+1}} |L^* f(x)|^p d\mu(x) \\
& \geq \sum_{s=[2\alpha_k/3]}^{m_k} \int_{I_{2s} \setminus I_{2s+1}} \left| L_{q_{\alpha_k}^s} f(x) \right|^p d\mu(x) \\
& \geq c \sum_{s=[2\alpha_k/3]}^{m_k} \int_{I_{2s} \setminus I_{2s+1}} \left(\frac{M_{2\alpha_k-1}^{1/p-1}}{\alpha_k^{3/2}} s M_{2s} \right)^p d\mu(x) \\
& \geq c \sum_{s=[2\alpha_k/3]}^{m_k} \frac{M_{2\alpha_k-1}^{1-p}}{\alpha_k^{p/2}} M_{2s}^{p-1} \\
& \geq \begin{cases} \frac{2^{\alpha_k(1-p)}}{\alpha_k^{p/2}}, & \text{when } 0 < p < 1, \\ c\sqrt{\alpha_k}, & \text{when } p = 1, \end{cases} \rightarrow \infty, \quad \text{when } k \rightarrow \infty.
\end{aligned}$$

Theorem 3 is proved.

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